

Computing the spectrum of black hole radiation in the presence of high frequency dispersion: an analytical approach

Steven Corley

*Department of Physics, University of Maryland
College Park, MD 20742-4111, USA
corley@physics.umd.edu¹*

Abstract

We present a method for computing the spectrum of black hole radiation of a scalar field satisfying a wave equation with high frequency dispersion. The method involves a combination of Laplace transform and WKB techniques for finding approximate solutions to ordinary differential equations. The modified wave equation is obtained by adding a higher order derivative term suppressed by powers of a fundamental momentum scale k_0 to the ordinary wave equation. Depending on the sign of this new term, high frequency modes propagate either superluminally or subluminally. We show that the resulting spectrum of created particles is thermal at the Hawking temperature, and further that the out-state is a thermal state at the Hawking temperature, to leading order in k_0 , for either modification.

¹current email address: scorley@phys.ualberta.ca

1 Introduction

Since Hawking's discovery that black holes radiate a thermal spectrum, [1], various other derivations of this effect have appeared, [2, 3, 4]. All seem to depend in some crucial way on the very high frequency behavior of the theory. Clearly such a derivation cannot be trusted without taking into account backreaction effects from the spacetime metric. This has led a number of authors, [5, 6, 7, 8], to consider the effects of high frequency dispersion on the Hawking spectrum. All have found, in the context of specific models, that the Hawking radiation remains almost exactly thermal at the Hawking temperature.

In this paper we show how the leading order contribution to the Hawking flux can be obtained by analytical methods for two models containing high frequency dispersion, one in which the high frequency modes propagate subluminally and one where the high frequency modes propagate superluminally. The method involves a combination of WKB and Laplace transform techniques to solve the modified wave equations, similar techniques were also used by Brout, Massar, Parentani, and Spindel [6] for different subluminal type models. We show for both models that the Hawking flux remains exactly thermal at the Hawking temperature to leading order in inverse powers of k_0 . We further show, to leading order in k_0 , that static observers far outside the black hole see the in-vacuum as a thermal state at the Hawking temperature, also in agreement with the ordinary wave equation case [9].

The specific models of high frequency dispersion considered in this paper are obtained by adding a higher derivative term, suppressed by a new fundamental momentum cutoff k_0 , to the ordinary wave equation with the appropriate sign to generate either subluminal or superluminal propagation of high frequency modes. The low frequency modes however behave as in the ordinary wave equation. Various subluminal theories have already been considered in [5, 6, 7, 8], in particular the subluminal equation considered in this paper is the same as considered in [8]. Recently Unruh [10] has also considered a superluminal modification (although different from the one considered here) to the ordinary wave equation. He has shown by numerically solving the modified wave equation that the spectrum is very nearly thermal at the Hawking temperature, and that the Hawking particles arise from vacuum fluctuations inside the horizon. Since these vacuum fluctuations would in principle evolve out of the singularity, a boundary condition at the singularity would be required. To avoid this problem, Unruh instead demanded that the vacuum fluctuations were in the ground state outside the singularity. In this paper we consider a similar superluminal modification to the ordinary wave equation, with the same type of boundary condition.

The remainder of this paper is as follows. We begin by introducing the model in section 2. In section 3 we discuss the method used to compute the particle creation in this model. Sections 4 and 5 then describe how the relevant solutions for particle creation are obtained for the subluminal and superluminal dispersion relations respectively, and

in section 6 we end with some conclusions. We use units with $c = \hbar = 1$.

2 Model

We consider a real scalar field propagating in a 2-dimensional black hole spacetime with metric

$$ds^2 = -dt^2 + (dx - v(x)dt)^2. \quad (1)$$

This is a generalization of the Lemaître line element of Schwarzschild spacetime where $v(x) = -\sqrt{2M/x}$. We follow the convention that $v(x) \leq 0$ and (in units where $c = 1$) the horizon is located at $v(x_h) = -1$. The action for the field is given by

$$S = \frac{1}{2} \int d^2x \left[((\partial_t + v\partial_x)\psi)^2 + \psi \hat{F}(\partial_x)\psi \right]. \quad (2)$$

To motivate this action we note that the black hole defines a preferred frame, the frame of freely falling observers. In the Lemaître coordinate system, $(\partial_t + v(x)\partial_x)$ is the unit tangent to free fall observers who start from rest at infinity, and ∂_x is its unit, outward pointing normal. Our action comes from modifying the derivative operator only along the unit normal ∂_x .

In the ordinary, minimally coupled action, $\hat{F}(\partial_x) = \partial_x^2$. In this paper we take

$$\hat{F}_{\pm}(\partial_x) = \partial_x^2 \pm \frac{1}{k_0^2} \partial_x^4. \quad (3)$$

The essential difference between these two derivative operators is that \hat{F}_+ produces subluminal propagation of high frequency modes² whereas \hat{F}_- produces superluminal propagation. The simplest way to see this behavior is to look at the respective dispersion relations. Varying the action produces the equations of motion

$$(\partial_t + \partial_x v)(\partial_t + v\partial_x)\phi = \partial_x^2 \phi \pm \frac{1}{k_0^2} \partial_x^4 \phi. \quad (4)$$

Assuming for simplicity that $v(x)$ is constant, we solve (4) by looking at mode solutions of the form

$$\phi(t, x) = \exp(i(\omega t - kx)). \quad (5)$$

This produces the dispersion relations

$$(\omega - vk)^2 = k^2 \mp (k/k_0)^4. \quad (6)$$

²Actually \hat{F}_+ produces both subluminal and superluminal propagation, however, the superluminal behavior does not play a crucial role in the analysis of this paper. It could in fact be removed by considering a model like Unruh's, [5].

In figure 1 we plot the square root of (6), that is we plot $(\omega - vk)$ and $\sqrt{k^2 \pm k^4/k_0^2}$ as functions of k for a fixed ω , along with the square root of the dispersion relation resulting from the ordinary wave equation. The intersection points are the allowed wavevector roots to (6). The value of the slope of the $\sqrt{k^2 \pm k^4/k_0^2}$ curve evaluated at an intersection point is the locally measured (by a freely falling observer) group velocity of a wavepacket centered about that wavevector. From figure 1 we see for the $\sqrt{k^2 + k^4/k_0^2}$ (corresponding to \hat{F}_-) curve that this slope is approximately one when $k \ll k_0$ and increases with increasing k , therefore the high frequency modes propagate superluminally. Similarly the $\sqrt{k^2 - k^4/k_0^2}$ (corresponding to \hat{F}_+) curve has slope approximately one when $k \ll k_0$ and decreases with increasing k until at some finite k it goes to zero, with a further increase in k the magnitude of the slope increases and eventually becomes much larger than one. Therefore the “large” wavevector modes propagate subluminally, but the “very large” wavevector modes, i.e., with k near k_0 , propagate superluminally. As already mentioned, the latter superluminal behavior is not essential for what we discuss in this paper, it could be removed by considering a dispersion relation like Unruh’s [5] in which the slope of the dispersion curve asymptotes to zero as k goes to infinity.

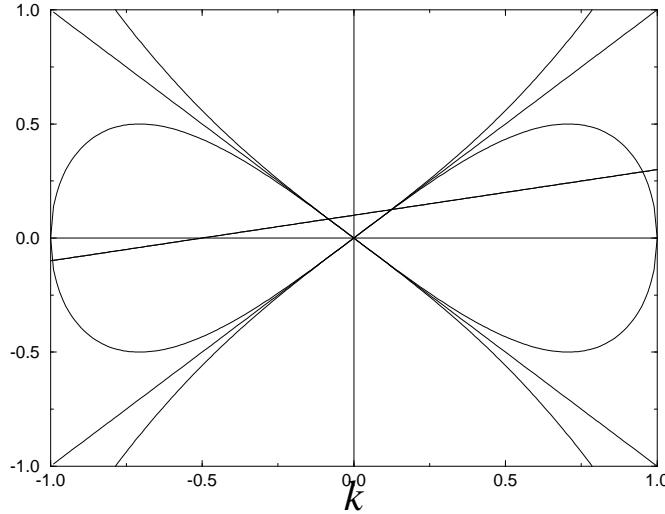


Figure 1: Plot of the dispersion relations for the ordinary wave equation and its subluminal and superluminal modifications. The intersection points with the $(\omega - vk)$ line are the possible wavevector roots. For the particular v shown, there are four real wavevector roots for the subluminal dispersion relation, but only two real roots for the superluminal case. The other two roots in this case are complex.

An important property of the action (2), when generalized to a complex scalar field,

is that it is invariant under constant phase transformations of the field. This leads to a conserved current j^μ . The time component j^0 , when integrated over a spatial slice, serves as a conserved inner product when evaluated on solutions to the equation of motion (4). For the metric (1), the inner product takes the form

$$(F, G) = i \int dx \left(F^* (\partial_t + v \partial_x) G - G (\partial_t + v \partial_x) F^* \right), \quad (7)$$

where $F(t, x)$ and $G(t, x)$ are solutions to (4). Two classes of solutions to the field equation (4) are of interest. The first are the positive free fall frequency wavepackets. They can be written as a sum of solutions satisfying

$$(\partial_t + v \partial_x) F(t, x) = -i\omega' F(t, x) \quad (8)$$

where $\omega' > 0$. The second are the positive Killing frequency wavepackets. These are a sum of solutions of the form $e^{-i\omega t} f(x)$ where $\omega > 0$. A positive free fall frequency wavepacket need not have a positive norm under (7) in general, but does when $v(x) = \text{constant}$. A positive Killing frequency wavepacket also need not have positive norm in general, but does when $v(x) = 0$.

To quantize the field we assume that $\hat{\phi}(x)$ is a self-adjoint operator solution to the field equation which satisfies the canonical commutation relations. We define for a normalized positive free fall frequency solution $f(t, x)$ the annihilation operator $a(f)$ by

$$a(f) \equiv (f, \hat{\phi}). \quad (9)$$

We make a similar definition for the annihilation and creation operators for a normalized positive Killing frequency solution $g(t, x)$ (assuming it is also positive norm).

3 Computing the Particle Production

The standard method of computing the amount of particle production in a given wavepacket is to propagate this wavepacket back in time to the hypersurface where the quantum state boundary condition is defined. In our case we are interested in computing the particle production in a late time, outgoing (right-moving), positive Killing frequency wavepacket. We assume that the state of the field is the free fall vacuum, which is defined by $a(p)|\text{ff}\rangle = 0$ for all positive free fall frequency modes p on the early time hypersurface. Denoting our late time wavepacket by ψ_{out} , one may show that its number expectation value in the free fall vacuum is

$$N(\psi_{out}) = \langle \text{ff} | a^\dagger(\psi_{out}) a(\psi_{out}) | \text{ff} \rangle = -(\psi_-, \psi_-) \quad (10)$$

where ψ_- is the negative free fall frequency part of ψ_{out} after being propagated back to the early time hypersurface (see [8] for a detailed derivation).

Rather than solving the full equation of motion, (4), we shall instead restrict ourselves to mode solutions of the form

$$\psi(t, x) = e^{-i\omega t} \phi(x). \quad (11)$$

Substituting into (4) (and setting $k_0 = 1$) produces the ordinary differential equation (ODE)

$$\pm \phi^{(iv)}(x) + (1 - v^2(x))\phi''(x) + 2v(x)(i\omega - v'(x))\phi'(x) - i\omega(i\omega - v'(x))\phi(x) = 0 \quad (12)$$

where the \pm refers to \hat{F}_\pm respectively and we have used a prime (\prime) to denote a derivative with respect to x . Restricting ourselves to mode solutions (11) therefore has the advantage that we need only solve an ODE.

To determine the boundary conditions for (12) it is necessary to study wavepacket propagation in these models. For the subluminal, \hat{F}_+ , equation this has been discussed in great detail in [6, 8]. The conclusion is that the late time, positive Killing frequency packet comes from a pair of ingoing, short wavelength packets, located far outside the black hole, and nothing³ else. In particular, no part of the wavepacket piles up against the horizon as with the ordinary wave equation, and furthermore nothing comes from across the horizon. This led [8] to conclude that the boundary condition for (12) in the subluminal case is that the solution decays across (and inside) the horizon⁴.

For the superluminal, \hat{F}_- , equation we refer the reader to [10, 11] for the details of wavepacket propagation. The conclusion is that the late time, positive Killing frequency packet comes from a pair of right-moving, short wavelength packets located far *inside* the horizon. This is hardly surprising given that the \hat{F}_- dispersion relation produces superluminal wave propagation. The important point concerning the boundary conditions though is that the only packet ever outside the horizon is the late time, positive Killing frequency packet⁵. This leads to the boundary conditions for the superluminal ODE that for $x \gg 0$ the solution reduces to a single mode with wavevector corresponding to the outgoing wavepacket.

Once we have solutions to the ODE's satisfying the boundary conditions just discussed, we may easily extract the particle creation. In the subluminal case, the solution at $x \gg 0$

³There could also be an ingoing, long wavelength packet which arises from scattering as the outgoing, positive Killing frequency packet is propagated back in time. For a slowly varying background spacetime we expect that the amount of scattering is negligible, which is indeed what we find.

⁴The solution cannot vanish inside the horizon because the coefficient of the highest derivative term does not vanish, as it does with the ordinary wave equation.

⁵This is not quite correct. Propagating the outgoing, positive Killing frequency packet back in time, a piece of it will scatter into an early time, ingoing packet. We again expect that the amount of scattering is very small for a slowly varying background spacetime, and in the leading order approximation discussed below we can ignore it. This problem can be dealt with by instead looking for mode solutions which correspond to propagating positive free fall frequency packets forward in time, which are necessary for computing the full quantum state. This will be discussed in subsection 5.1.

(where $v(x) \approx \text{constant}$) can be decomposed as

$$\phi(x) = \sum_{l=1}^4 c_l(\omega) e^{ik_l(\omega)x}, \quad (13)$$

where $k_l(\omega)$ are the roots to the subluminal dispersion relation (6). From figure 1 it is easy to see that two of these roots are positive and the other two are negative. The late time wavepacket corresponds to the small, positive wavevector (k_{+s}), and the early time, ingoing wavepackets to the large, positive wavevector (k_+) and the large, negative wavevector (k_-) respectively. The small negative wavevector (k_{-s}) corresponds to a long wavelength, ingoing wavepacket which will not be important in this leading order calculation. The number expectation value for a mode of Killing frequency ω is then

$$N(\omega) = \frac{|\omega'(k_-)v_g(k_-)c_-^2(\omega)|}{|\omega'(k_{+s})v_g(k_{+s})c_{+s}^2(\omega)|}. \quad (14)$$

The kinematic factors $v_g(k)$ and $\omega'(k)$ are the group velocity as measured by a static observer and the frequency as measured by a freely falling observer of a wavepacket narrowly peaked about wavevector k .

The superluminal equation can be handled in almost exactly the same manner. Far outside the horizon, where $v(x)$ is approximately constant and satisfies $0 > v(x) > -1$, it is again easy to see from figure 1 that there is one positive wavevector root (k_{+s}) of the dispersion relation (6) (with the plus sign) and one negative wavevector root (k_{-s}). Our boundary conditions dictate that the solution at $x \gg 0$ is⁶

$$c_{+s} e^{ik_{+s}(\omega)x}. \quad (15)$$

To avoid dealing with the singularity, we shall also assume that $v(x)$ becomes constant behind the horizon. When the slope of the straight line in figure 1 is larger than one, it is easy to see that there is one positive wavevector root (k_+) to the dispersion relation (6) (with the positive sign) and three negative wavevector roots which we denote as k_{-s} , k_{-m} , and k_- in order of increasing magnitude (s denoting *small* and m denoting *middle*). As we shall see below, only the large positive and large negative wavevector solutions will contribute to the solution. It follows that the solution in this region is of the form

$$c_+ e^{ik_+(\omega)x} + c_- e^{ik_-(\omega)x}. \quad (16)$$

The number expectation value in this case again becomes (14), with the kinematic factors appropriate for the superluminal equation, \hat{F}_- .

⁶Note that we use the same notation for denoting the wavevectors for both the subluminal and superluminal cases; however, the actual values of the wavevectors for a given ω differ between the two cases.

4 Approximate solutions to the subluminal equation

The methods applied to find approximate solutions to ODE (12) are the same as those used in solving the Schrodinger equation for a tunneling potential. For the general potential one may find approximate solutions to the Schrodinger equation by the WKB method [14]; however, about a classical turning point (i.e., where the kinetic energy vanishes) the WKB approximation breaks down. Approximate solutions can nevertheless be obtained by expanding the potential $V(x)$ in the full Schrodinger equation about the classical turning point x_{tp} . Solutions to the resulting equation are straightforward to find and are valid in the region

$$|x - x_{tp}|^{n-1} \ll |n!V'(x_{tp})/V^{(n)}(x_{tp})| \quad (17)$$

if $V''(x_{tp}), \dots, V^{(n-1)}(x_{tp})$ all vanish.

If the WKB solutions are valid in regions on either side of the classical turning point which overlap with the region of validity of the solution to the linearized potential equation, then we can obtain approximate solutions to the full Schrodinger equation over the entire range of x . We now apply this method to ODE (12) for the subluminal equation, \hat{F}_+ . We begin by finding the approximate solution about the horizon relevant for particle creation in subsection 4.1, and then match this solution to WKB solutions outside the horizon in subsection 4.2. From this solution we then compute the amount of particle creation in subsection 4.3. Finally we show in subsection 4.4 that this method can be extended to computing the complete out-state to leading order in k_0 , i.e., compared to just computing number expectation values.

4.1 Approximate solutions about the horizon

In words the calculations we present in this subsection are as follows. We find approximate solutions to ODE (12) about the horizon by linearizing the x -dependent coefficients in the equation and solving the resulting equation by the method of Laplace transforms. These solutions are given by contour integrals in the complex s -plane (where s is the Laplace transform variable). The contour C_0 (see figure 2) corresponding to the x -space solution inside the horizon is chosen so that the boundary conditions described in section 3 are satisfied, i.e., that the solution decays inside the horizon, as these are the boundary conditions relevant for particle creation. The x -space solution outside the horizon must then arise from a contour that is deformable to C_0 . This contour is broken up into three separate contours: C_1 , C_2 , and C_3 (see figure 3). By comparing the solutions corresponding to these contours to the WKB solutions (computed in subsection 4.2) of ODE (12), we show that C_3 corresponds to the late time, outgoing Hawking particle and that C_1 and C_2 correspond to the ingoing, large positive and negative wavevector packets respectively from which the outgoing Hawking particle arises. Once this has been done it is a simple matter to compute the particle creation as discussed in section 3.

We first linearize $v(x)$ and $v'(x)$ about the horizon as

$$v(x) \approx -1 + \kappa x \quad (18)$$

$$v'(x) \approx \kappa + \kappa_1^2 x \quad (19)$$

where κ is the surface gravity of the black hole described by the metric (1) and κ_1 is a higher order correction to $v(x)$. Substituting into (12) and keeping only linear terms in x yields

$$\phi^{(iv)}(x) + 2\kappa x \phi''(x) + 2(-(i\omega - \kappa) + (\kappa(i\omega - \kappa) + \kappa_1^2)x)\phi'(x) - i\omega(i\omega - \kappa - \kappa_1^2 x)\phi(x) = 0. \quad (20)$$

Validity of this equation requires that $|\kappa x| \ll 1$ and $|\kappa_1^2 x / \kappa| \ll 1$. To leading order we may therefore further simplify the equation as

$$\phi^{(iv)}(x) + 2\kappa x \phi''(x) - 2(i\omega - \kappa)\phi'(x) - i\omega(i\omega - \kappa)\phi(x) = 0. \quad (21)$$

This is the equation we shall use in this paper to find approximate solutions about the horizon; however, to compute correction terms to the flux, we must keep at least all linear terms in x and possibly even higher order terms in x . We shall discuss this further in the conclusions section.

We use the method of Laplace transforms, [15, 16], to solve (21). Writing the solution as a Laplace transform,

$$\phi(x) = \int_C ds e^{sx} \tilde{\phi}(s), \quad (22)$$

(where C is the contour of integration) and substituting into (21) yields the s -space⁷ ODE

$$\partial_s \left(\ln(s^2 \tilde{\phi}(s)) \right) = \frac{s^4 - 2(i\omega - \kappa)s - i\omega(i\omega - \kappa)}{2\kappa s^2}. \quad (23)$$

Equation (23) is easily solved as

$$\tilde{\phi}(s) = s^{-1-i\omega/\kappa} \exp \left(\frac{1}{2\kappa} \left(\frac{s^3}{3} + \frac{i\omega(i\omega - \kappa)}{s} \right) \right) \quad (24)$$

To obtain the x -space solution, we substitute $\tilde{\phi}(s)$ into (22) and integrate. The choice of contour C over which we integrate is dictated by the boundary conditions discussed in section 3, i.e., we want $\phi(x)$ to decay inside the horizon. Before finding the appropriate contour to produce this behavior let's first understand the generic properties that the contour must satisfy. Specifically, note that $\tilde{\phi}(s)$ is dominated at large $|s|$ by the $\exp(s^3/(6\kappa))$ term, and therefore for the integral to converge (assuming the contour C runs to infinity,

⁷We have dropped a boundary term to obtain (23). This term will vanish by our choice of contours below.

which it need not) the contour must asymptote to a region where $Re(s^3) < 0$ since κ is real and positive. Writing $s = re^{i\theta}$, this implies that the contour must asymptote to any of the three regions

$$\begin{aligned} \text{Region 1} &\leftrightarrow \frac{\pi}{6} < \theta < \frac{\pi}{2} \\ \text{Region 2} &\leftrightarrow \frac{5\pi}{6} < \theta < \frac{7\pi}{6} \\ \text{Region 3} &\leftrightarrow \frac{3\pi}{2} < \theta < \frac{11\pi}{6}. \end{aligned} \tag{25}$$

In figure 2 these appear as the unmarked regions.

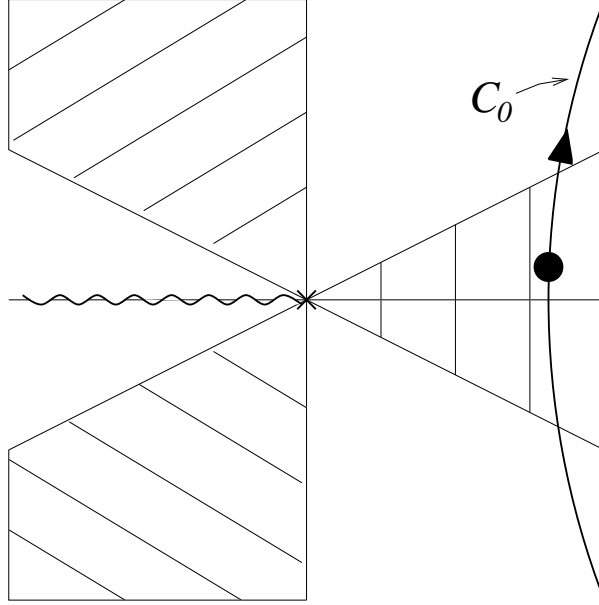


Figure 2: Diagram of the steepest descent contour C_0 . The unmarked regions are directions in which the contour must asymptote for the integral to converge. The \times 's are singularities of the integrand and the wavy line is a branch cut.

To evaluate the contour integral (22) we first consider the $x < 0$ case. We must choose a contour that yields a solution that decays with decreasing x inside the horizon, and that asymptotes to any of the three regions (25). To find this solution we approximate the contour integral (22) by the method of steepest descents, [16].

First rewrite the contour integral as

$$\phi(x) = \int_C ds g(s) e^{x f(s)} \tag{26}$$

where

$$g(s) = s^{-1-i\omega/\kappa} \quad (27)$$

and

$$f(s) = s + \frac{1}{2\kappa x} \left(\frac{s^3}{3} + \frac{i\omega(i\omega - \kappa)}{s} \right). \quad (28)$$

To evaluate (26) by steepest descents we first locate the saddle points of $f(s)$. These are given by the roots of $df(s)/ds = 0$, [16], which in this case are approximated by

$$s_{\pm} \approx \pm \sqrt{-2\kappa x} \quad (29)$$

(since we are in a region where $\omega, \kappa \ll |\kappa x|$). The contours of steepest descent through these saddle points are given by $Im(f(s) - f(s_{\pm})) = 0$ and $Re(x(f(s) - f(s_{\pm}))) < 0$. Using this one may show that the direction of the steepest descent contours through s_+ and s_- are $-\pi/2$ to $\pi/2$ and 0 to π respectively. It is not hard to show that the steepest descent contour, C_0 , through s_+ asymptotes to regions 1 and 3 as shown in figure 2. The contour integral in this case is now obtained by the standard formula

$$\phi_0(x) \approx g(s_+) \sqrt{\frac{2\pi}{|xf''(s_+)|}} e^{xf(s_+) + i\alpha_+} \quad (30)$$

where $\alpha_+ = \pi/2$ if we traverse the contour in the direction indicated in figure 2. To lowest order in ω and κ this reduces to

$$\phi_0(x) \approx -\sqrt{2\pi\kappa} (-2\kappa x)^{-3/4-i\omega/(2\kappa)} \exp\left(-\frac{2}{3}\sqrt{2\kappa}|x|^{3/2}\right). \quad (31)$$

We immediately see that this is exponentially decaying with decreasing x (recall that $x < 0$). The contour through s_- produces an exponentially growing solution, hence our desired solution is given by the contour C_0 . Finally, note that $g(s)$ is singular at $s = 0$, and that we must choose a branch cut from this point. We choose the branch cut to run along the negative real s -axis.

At this point the reader may be wondering about the validity of the approximations made so far. The steepest descents method requires that $|x| \gg 1$ while validity of the approximate ODE (21) requires that $|\kappa x| \ll 1$. As long as $\kappa \ll 1$, there is always a wide range of x values satisfying both conditions, i.e., $1 \ll |x| \ll 1/\kappa$. Such κ correspond to black hole temperatures $T_H \ll 1$ (or $T_H \ll k_0$ if we restore k_0). For example, if the Planck length is also one in these units (i.e., $k_0 = 1 = 1/l_{Pl}$), and κ is the surface gravity of a solar mass black hole, the inequality on x becomes $1 \ll |x| \ll 10^{38}$. Clearly there is no problem in satisfying this inequality. It is convenient to keep these numbers in mind for later approximations.

Now we turn to evaluating $\phi(x)$ for $x > 0$. In principle we must evaluate the contour integral (26) over the same contour as in the $x < 0$ case, i.e., C_0 . However, by Cauchy's

theorem we may deform the contour (keeping the endpoints fixed) through any region in which the integrand is analytic to a new contour, hopefully one where the integral is easier to evaluate. In particular we may deform the contour so that it runs through any nearby saddle points so that we may again approximate the integral by the method of steepest descents. In fact most of the work for these saddle points has already been done. They are still given by (29) except that now $x > 0$ and therefore are both imaginary (to leading order), i.e., $s_{\pm} \approx \pm i\sqrt{2\kappa x}$. The direction of the steepest descent contours through s_- and s_+ are now given by $7\pi/4$ to $3\pi/4$ and $5\pi/4$ to $\pi/4$ respectively. From this one can easily see that the steepest descent contour, C_1 , through s_- asymptotes to regions 2 and 3 (25) and the steepest descent contour, C_2 , through s_+ asymptotes to regions 1 and 2 (25), as shown in figure 3. Evaluating the leading order contributions to these contour integrals as before results in

$$\phi_1(x) \approx e^{i5\pi/4} e^{-\pi\omega/(2\kappa)} \sqrt{2\pi\kappa} (2\kappa x)^{-3/4-i\omega/(2\kappa)} \exp\left(-i\frac{2}{3}\sqrt{2\kappa x}^{3/2}\right) \quad (32)$$

$$\phi_2(x) \approx e^{-i\pi/4} e^{\pi\omega/(2\kappa)} \sqrt{2\pi\kappa} (2\kappa x)^{-3/4-i\omega/(2\kappa)} \exp\left(i\frac{2}{3}\sqrt{2\kappa x}^{3/2}\right) \quad (33)$$

where we have chosen the directions of the contours as depicted in figure 3.

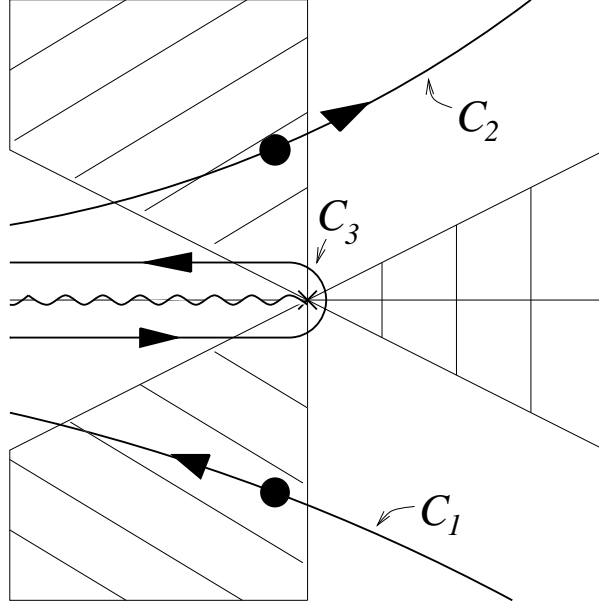


Figure 3: Diagram of the steepest descent contours C_1 , C_2 , and C_3 . C_1 and C_2 pass through the saddle points s_+ and s_- respectively. The unmarked regions are directions in which the contour must asymptote for the integral to converge. The \times 's are singularities of the integrand and the wavy line is a branch cut.

The contour $C_1 + C_2$ is not by itself deformable to C_0 , but if we add in the contour C_3 (see figure 3) which asymptotes to region 2 on either side of the branch cut, then $C_1 + C_2 + C_3$ is deformable to C_0 . To evaluate the contour integral over C_3 , first define the new integration variable $e^{-i\pi}t := sx$. This produces

$$\phi_3(x) = x^{i\omega/\kappa} \int_{\bar{C}_3} dt (-t)^{-1-i\omega/\kappa} \exp\left(-t + \left(-\frac{t^3}{6\kappa x^3} - \frac{i\omega(i\omega - \kappa)x}{2\kappa t}\right)\right) \quad (34)$$

where the new contour \bar{C}_3 runs from infinity just above the positive real axis, counter-clockwise about the origin, and back to infinity just below the positive real axis. Ignoring the t^3 and t^{-1} terms in the exponent for the moment, we note that the remainder is just a gamma function, i.e., using the integral representation [17],

$$\Gamma(\nu) = -\frac{1}{i2\sin(\pi\nu)} \int_{\bar{C}_3} dt (-t)^{-1+\nu} e^{-t} \quad (35)$$

we arrive at

$$\phi_3(x) \approx -2 \sinh(\pi\omega/\kappa) \Gamma(-i\omega/\kappa) x^{i\omega/\kappa}. \quad (36)$$

To see when this approximation holds, expand

$$\exp\left(-\frac{t^3}{6\kappa x^3}\right) \approx 1 - \frac{t^3}{6\kappa x^3} + O\left(\frac{t^6}{\kappa^2 x^6}\right). \quad (37)$$

Evaluating the integral (34) with the t^3 term produces the correction term to $\phi_3(x)$,

$$\delta\phi_3(x) = \frac{\sinh(\pi\omega/\kappa)}{3\kappa x^3} \Gamma(-i\omega/\kappa) x^{i\omega/\kappa}. \quad (38)$$

Using the identity $\Gamma(z+1) = z\Gamma(z)$ one may show that $|\delta\phi_3(x)/\phi_3(x)| \ll 1$ holds if

$$1 \ll |\kappa x^3|. \quad (39)$$

If we expand $\exp(-i\omega(i\omega - \kappa)x/(2\kappa t))$ in the same way and evaluate the leading order correction term to $\phi_3(x)$ as before, we find that we need

$$|\omega x| \ll 1. \quad (40)$$

When these conditions hold, $\phi_3(x)$ is well approximated by (36) (in the asymptotic expansion sense).

To summarize, we have found an approximate solution $\phi(x)$ to the mode equation (12) satisfying the boundary conditions that it decay inside the horizon. Just outside the horizon the solution is given by $\phi(x) = \phi_1(x) + \phi_2(x) + \phi_3(x)$ (see (32), (33), and (36) respectively). We now propagate this solution out to a region where $v(x)$ is essentially constant by patching onto WKB solutions which are valid outside the horizon. Knowing the solution in the constant $v(x)$ region will then allow us to easily extract the particle flux.

4.2 WKB solutions

To find approximate solutions to the mode equation (12) by the WKB method, assume a solution of the form

$$\phi(x) = e^{i \int dx k(x)} \quad (41)$$

where the wavevector $k(x)$ is an unknown function of x . Substitution yields

$$\begin{aligned} k^4 - (1 - v^2)k^2 - 2v\omega k + \omega^2 = \\ i \frac{d}{dx} (2k^3 - (1 - v^2)k - \omega v) + (4kk'' + 3(k')^2) - ik''' \end{aligned} \quad (42)$$

where we denote derivatives with respect to x by primes (\prime). If $v(x)$ is a slowly varying function of x , then we expect $k(x)$ also to be slowly varying. We therefore try to solve for k perturbatively in derivatives of $v(x)$. More precisely, let $x \rightarrow \alpha x$ (we will take $\alpha = 1$ at the end), then (42) becomes

$$\begin{aligned} k^4 - (1 - v^2)k^2 - 2v\omega k + \omega^2 = \\ \frac{i}{\alpha} \frac{d}{dx} (2k^3 - (1 - v^2)k - \omega v) + \frac{1}{\alpha^2} (4kk'' + 3(k')^2) - \frac{i}{\alpha^3} k'''. \end{aligned} \quad (43)$$

In this equation we see on the right-hand-side that derivatives of $k(x)$ and $v(x)$ with respect to x are suppressed by powers of $1/\alpha$.

Now assume that $k(x)$ may be expanded in inverse powers of α as

$$k(x) = k^{(0)}(x) + \frac{1}{\alpha} k^{(1)}(x) + \dots \quad (44)$$

Substituting into (43) and demanding that the coefficients of each power of $1/\alpha$ separately vanish produces an infinite set of equations, the lowest orders being

$$(k^{(0)})^4 - (1 - v^2)(k^{(0)})^2 - 2v\omega k^{(0)} + \omega^2 = 0 \quad (45)$$

$$k^{(1)} = \frac{i}{2} \frac{d}{dx} \ln(2(k^{(0)})^3 - (1 - v^2)k^{(0)} - v\omega). \quad (46)$$

Although the leading order equation for $k^{(0)}$, (45), can be solved exactly producing a set of four wavevectors, the expressions are quite unwieldy. Fortunately, since we are mainly interested in Killing frequencies ω satisfying $\omega \ll 1$, we only need find approximate wavevector roots to this equation. Once these roots are known, the $1/\alpha$ corrections to them can be found by substituting the respective wavevector root $k^{(0)}$ into (46) and solving for $k^{(1)}$. These computations produce the wavevectors

$$k_{\pm} = \pm \sqrt{1 - v^2} + \frac{\omega v}{1 - v^2} + i \frac{3}{4} \frac{d}{dx} \ln(1 - v^2) + \mathcal{O}(\omega^2) \quad (47)$$

$$k_{+s} = \frac{\omega}{1 + v} + \mathcal{O}(\omega^3) \quad (48)$$

$$k_{-s} = -\frac{\omega}{1 - v} + \mathcal{O}(\omega^3), \quad (49)$$

where we have set $\alpha = 1$. The corresponding WKB solutions are

$$\phi_{\pm}(x) \approx (1 - v(x)^2)^{-3/4} e^{\pm i \int dx \sqrt{1-v(x)^2}} e^{i\omega \int dx v(x)/(1-v^2(x))} \quad (50)$$

$$\phi_{+s}(x) \approx e^{i\omega \int dx/(1+v(x))} \quad (51)$$

$$\phi_{-s}(x) \approx e^{-i\omega \int dx/(1-v(x))}. \quad (52)$$

The condition of validity for these approximate solutions is that $|k^{(1)}(x)/k^{(0)}(x)| \ll 1$. For the k_{\pm} wavevectors this ratio is

$$\left| \frac{k_{\pm}^{(1)}(x)}{k_{\pm}^{(0)}(x)} \right| \approx \frac{3}{2} \left| \frac{v(x)v'(x)}{(1-v^2(x))^{3/2}} \right|. \quad (53)$$

We are interested in $v(x)$'s containing black holes. The horizon of a black hole in these units is located at $v(x_h) = -1$, therefore the right-hand-side of (53) clearly becomes arbitrarily large as we approach the horizon (assuming that $v'(x) \neq 0$ which are the only cases we consider here). It follows that the WKB approximation will break down around the horizon. Far from the horizon (and outside the black hole) $v(x)$ asymptotes to a constant $-1 < v_0 < 0$ and $v'(x)$ goes to zero, therefore the WKB approximation will be valid. For the k_{+s} mode a ratio similar to (53) holds, and therefore the WKB approximation again fails around the horizon, but is valid far outside of it. To compute this ratio, we must compute k_{+s} to order $O(\omega^3)$, however since we will not need this later we do not give the explicit expressions here.

4.3 The spectrum

We now have all the ingredients necessary to compute the leading order spectrum of black hole radiation. First note that validity of the WKB approximation (53) requires that $1 \ll \sqrt{\kappa x^3}$. Furthermore the approximate solution from the Laplace transform method is valid when $1 \ll |x| \ll 1/\kappa$. Since we are considering cases where $\kappa \ll 1$, there is always a region where both the WKB and Laplace transform solutions are valid.

If we evaluate the integrals appearing in the WKB solutions of (50), (51) and, (52) respectively in a region just outside the horizon where $v(x)$ is given by the linearized expression (18), we find that the solution outside the horizon obtained from the Laplace transform method can be expressed as

$$\begin{aligned} \phi(x) &= e^{-i\pi/4} \sqrt{2\pi\kappa} (-i e^{-\pi\omega/(2\kappa)} \phi_{-}(x) + e^{\pi\omega/(2\kappa)} \phi_{+}(x)) \\ &- e^{-\pi\omega/\kappa} (e^{2\pi\omega/\kappa} - 1) \Gamma(-i\omega/\kappa) \phi_{+s}(x). \end{aligned} \quad (54)$$

With $\phi(x)$ decomposed in terms of the WKB solutions, we are allowed to evaluate it at large x , i.e., where $v(x)$ is essentially constant. In this region the WKB solutions reduce

to simple modes (up to multiplicative constants) and we need only extract the coefficients of these modes in order to compute the particle creation rate as given in (14). A simple computation yields

$$N(\omega) = \frac{1}{e^{2\pi\omega/\kappa} - 1}, \quad (55)$$

exactly a thermal spectrum at the Hawking temperature $T_H = \kappa/(2\pi)$.

As a check on our results, recall that invariance of the action (2) under constant phase transformations of the field leads to the existence of a conserved current j^μ . When this current is evaluated on fixed Killing frequency mode solutions, the time component j^t is manifestly time independent and the conservation law reduces to $\partial_x j^x = 0$, i.e., j^x is constant. The exact form of j^x is complicated, but when evaluated on a mode of the form $c(\omega) \exp(-i\omega t + ik(\omega)x)$ in a region where $v(x)$ is constant, it reduces to

$$j^x(k(\omega)) = \omega'(k(\omega)) v_g(k(\omega)) |c(k(\omega))|^2. \quad (56)$$

For the solution of interest to us, i.e., the one that decays inside the horizon, the spatial part of the current must vanish everywhere. Using the solution given by (54), it is easy to show that this is indeed the case.

We have made a number of approximations in order to compute the leading order flux given by (55), we would now like to collect them to find out what restrictions they place on the allowed parameter ranges. First recall that we have restricted κ to values $\kappa \ll 1$ (in units of $k_0 = 1$). Physically this says that we only expect to get a thermal spectrum of radiation when the black hole is large, and therefore has a small temperature.

The range of validity of the Laplace transform solutions in the spatial variable x is also restricted. We have already seen that we need $|x| \gg 1$ for the steepest descents approximation to hold for the various contour integrals and $|\kappa x| \ll 1$ for the approximate ODE (21) to be valid. Closer investigation of the correction terms to the WKB solutions and the Laplace transform solutions shows that we need

$$1/\kappa^{1/3} \ll |x| \ll 1/\kappa^{3/5}. \quad (57)$$

This inequality means that the matching of the WKB and Laplace transform solutions can be done anywhere in this range. To derive these inequalities, note that the WKB and Laplace transform solutions cannot be matched to arbitrary order because in fact they solve different equations. The Laplace transform solutions were obtained by finding approximate solutions to the linearized $v(x)$ ODE (21), while the WKB solutions were obtained by finding approximate solutions to the full ODE. By computing the first set of correction terms to both the WKB and Laplace transform solutions that differ, i.e., do not match, and demanding that they are small (compared to the leading order terms), we derive the above inequality.

These correction terms also restrict the range of ω . Since the corrections are x dependent, by matching the WKB and Laplace transform solutions about an appropriate

x satisfying (57) we can minimize the difference between the solutions. To carry this out we compute the leading order correction terms for the WKB and Laplace transform solutions that differ and sum their absolute values. As a specific case, the leading order relative difference between the WKB solution corresponding to the wavevector k_+ and the Laplace transform solution corresponding to the contour C_2 is

$$\sigma \approx \left(\frac{2}{\pi^2}\right)^{1/4} \frac{\omega}{\kappa} \frac{\sqrt{\kappa}}{(2\kappa x)^{3/4}} + \frac{1}{40} \frac{(2\kappa x)^{5/2}}{\kappa} \quad (58)$$

in the limit of $\omega \gg \kappa$. Minimizing this with respect to x we find that $\kappa x_{min} \approx (9\kappa\omega^2/(4\pi))^2/13$ and

$$\sigma \approx \frac{13}{120} \left(\frac{2}{\pi^2}\right)^{5/26} 12^{10/13} \frac{\omega^{10/13}}{\kappa^{8/13}}. \quad (59)$$

Demanding that $\sigma \ll 1$ we arrive at the constraint

$$\omega \ll \frac{1}{12} \left(\frac{\pi^2}{2}\right)^{1/4} \left(\frac{120}{13}\right)^{13/10} \kappa^{4/5}. \quad (60)$$

A similar computation can also be carried out for the WKB and Laplace transform solutions corresponding to the k_- and k_{+s} wavevector roots. In the limit of $\omega \gg \kappa$ a weaker constraint than above is obtained from the k_{+s} root, and the same constraint as above is obtained for the k_- wavevector. In the same manner as above a lower bound on the allowed range of ω can be obtained. From the k_+ wavevector we find that we need

$$\omega \gg \frac{13}{6\pi} \left(\frac{2}{\pi}\right)^{5/13} \frac{1}{20^{3/13}} \kappa^{15/13}. \quad (61)$$

It follows that the range of validity of the analytical results presented above are bounded both above and below in the parameter ω .

4.4 Computing the quantum state

We have so far computed the outgoing flux of particles from a black hole for the subluminal equation of motion (4) (with the plus sign). There is, however, much more information contained in the quantum state than just number expectation values, for instance, correlations. It is therefore of interest to compute the full quantum state in this modified theory and compare it to the state arising with the ordinary wave equation, as already computed by Wald [9].

To be a bit more precise, we are not actually “computing” the quantum state because we already know what it is, i.e., we have assumed that it is the free fall vacuum. What we are going to do is re-express this state in terms of a vacuum state defined by late time observers. We define the out-Hilbert space as the tensor product of Hilbert spaces

on either side of the horizon. Outside the horizon we use Killing frequency to define the Hilbert space, as we have done so far. Inside the horizon we do the following. We take a $v(x)$ that asymptotes to a constant (less than -1), then we define our Hilbert space in this region using free fall frequency. If we are only interested in the observations made by the outside observer then we would trace over the inside degrees of freedom, in which case the Hilbert space we use inside the horizon is irrelevant. For the computations here though, the choice we have made is the simplest.

Our method of computing the state is very similar to the techniques employed by Wald [9]. In the time dependent picture we would take an ingoing, positive free fall frequency wavepacket and evolve it from the hypersurface where the free fall vacuum is defined to the hypersurface where the out vacuum is defined. If this packet is sufficiently peaked in its wavevector, we may follow it on the dispersion relation as discussed in detail in [8]. It is not hard to see that this packet will propagate toward the horizon and scatter (mode convert), the reflected piece propagates (forward in time) away from the horizon to the region where $v(x)$ is essentially constant, and the transmitted packet propagates deeper inside the horizon to where $v(x)$ is essentially constant. The late time packet outside the horizon corresponds to the Hawking particle, and the late time packet inside the horizon corresponds to the partner. If the initially positive free fall frequency packet $\psi_{+\text{ff}}$ contains only positive Killing frequencies, the final packet $\psi_{+\text{out}}$ outside the horizon will contain only positive Killing frequencies and the final packet $\psi_{-\text{in}}$ inside the horizon will contain only negative free fall frequencies. The annihilation operator associated with $\psi_{+\text{ff}}$, i.e., $a(\psi_{+\text{ff}}) := (\psi_{+\text{ff}}, \hat{\phi})$, annihilates the free fall vacuum. Using the time independence of the inner product, we therefore derive the equation

$$(a(\psi_{+\text{out}}) - a^\dagger(\psi_{-\text{in}}^*))|\text{ff}\rangle = 0. \quad (62)$$

Similarly, if the initially positive free fall frequency packet contains only negative Killing frequencies, the final packet outside the horizon will contain only negative Killing frequencies and the final packet inside the horizon will contain only positive free fall frequencies. Using this, a relation similar to (62) can be derived. As shown by Wald [9], given a complete set of relations like (62) (constructed by taking a complete set of ingoing, positive free fall frequency packets), we can re-express the free fall vacuum in terms of the out vacuum.

As we have done thus far, we shall actually use mode solutions instead of wavepackets. We derive the mode solutions which, when appropriately summed, produce the time dependent wavepacket solutions just discussed. We have already derived one mode solution in (54), although it is not of the form that we want. Rather it decays inside the horizon and is a superposition of plane waves with wavevectors k_{+s} , k_+ , and k_- far outside the horizon (where $v(x)$ is essentially constant). In the time dependent picture it corresponds to propagating a pair of ingoing, positive and negative free fall frequency wavepackets forward in time, with just the right relative weights so that the entire packet completely

mode converts around the horizon, turns around, and propagates out to the constant $v(x)$ region. To obtain the mode solutions that we want, we shall construct another mode solution below which is a superposition of plane waves with wavevectors k_+ and k_- far outside the horizon (where $v(x)$ is essentially constant), whereas inside the horizon (where $v(x)$ is again essentially constant) it reduces to a plane wave with wavevector k_- . In the time dependent picture this mode solution corresponds to propagating a pair of ingoing, positive and negative free fall frequency wavepackets forward in time, with just the right relative weights (although different than above) so that the entire packet propagates across the horizon and converts into just a negative free fall frequency packet. By adding these two mode solutions, call them the n-modes since they are the relevant ones for computing number expectation values, with the correct relative coefficient, we can eliminate the k_- (k_+) mode outside the horizon. These are the mode solutions we want, call them the s-modes since they are the relevant ones for computing the state, because they correspond to propagating an ingoing, positive (negative) free fall frequency packet forward in time which splits around the horizon into a pair of wavepackets, one propagates back away from the horizon, and the other falls inside the black hole.

Because the details of computing the other n-mode solution discussed above are essentially the same as discussed in subsections 4.1 and 4.2, we shall only sketch the computation. We first solve the mode equation (12) (with the plus sign) about the horizon by the method of Laplace transforms exactly as before. The only difference is that the contour of integration must be changed so as to satisfy the boundary conditions that the solution reduce to a plane wave with wavevector k_- inside the horizon where $v(x)$ is essentially constant. A straightforward computation shows that the contour C_4 shown in figure 4 does the job, i.e., evaluating the contour integral (26) over C_4 and propagating the solution deeper inside the black hole by the WKB approximation to the constant $v(x)$ region shows that the solution is

$$\phi_4(x) \approx 2e^{\pi\omega/\kappa} \sinh(\pi\omega/\kappa) \Gamma(-i\omega/\kappa) \phi_-(x), \quad (63)$$

which is the boundary condition we want.

To evaluate the solution outside the horizon, we deform the contour C_4 into C_2 and C_5 as shown in figure (4). These we can evaluate by the method of steepest descents. In fact C_2 is exactly the same as the contour C_2 before, see figure 3. C_5 is the same as C_1 before (see figure 3), except that it lies on a different Riemann sheet, so only the overall scale changes. The complete solution outside the horizon after being propagated out to the constant $v(x)$ region by the WKB approximation is

$$\phi_2(x) + \phi_5(x) \approx e^{-i\pi/4} e^{\pi\omega/(2\kappa)} \sqrt{2\pi\kappa} (\phi_+(x) - ie^{\pi\omega/\kappa} \phi_-(x)). \quad (64)$$

Combining this with $\phi_4(x)$ produces the connection formula

$$2e^{\pi\omega/\kappa} \sinh(\pi\omega/\kappa) \Gamma(-i\omega/\kappa) \phi_-(x) \leftrightarrow e^{-i\pi/4} e^{\pi\omega/(2\kappa)} \sqrt{2\pi\kappa} (\phi_+(x) - ie^{\pi\omega/\kappa} \phi_-(x)) \quad (65)$$

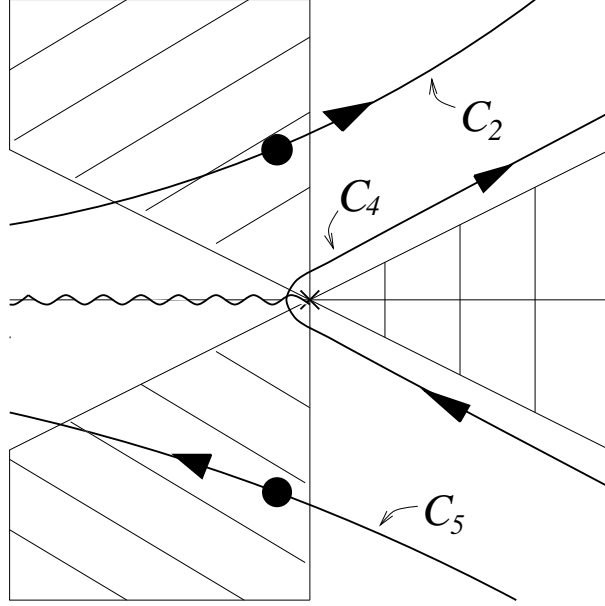


Figure 4: Diagram of the steepest descent contours C_2 , C_4 , and C_5 . C_2 and C_5 pass through the saddle points s_+ and s_- respectively, the solutions for these contours are valid for $x > 0$. The solution corresponding to the contour C_4 is valid for $x < 0$. The unmarked regions are directions in which the contour must asymptote for the integral to converge. The \times 's are singularities of the integrand and the wavy line is a branch cut.

where the left-hand-side refers to $x < 0$ and the right-hand-side to $x > 0$. This is the second of the n-mode solutions, the first is given in (54).

To obtain the s-mode solutions, we add the n-mode solutions given in (54) and (65) with the correct relative coefficient so that either the $\phi_-(x)$ mode or the $\phi_+(x)$ mode cancels at large positive x , this produces the connection formulae

$$e^{\pi\omega/\kappa}\phi_-(x) \leftrightarrow \phi_{+s}(x) + e^{-i3\pi/4} \frac{\sqrt{2\pi\kappa}}{\Gamma(-i\omega/\kappa)} \phi_-(x) \quad (66)$$

$$e^{-\pi\omega/\kappa}\phi_-(x) \leftrightarrow \phi_{+s}(x) - e^{-i\pi/4} e^{-3\pi\omega/(2\kappa)} \frac{\sqrt{2\pi\kappa}}{\Gamma(-i\omega/\kappa)} \phi_+(x) \quad (67)$$

where again the left-hand-side refers to negative x and the right-hand-side to positive x . Noting that the annihilation operator associated with the modes $\phi_+(x)$ and $\phi_-^*(x)$ at large positive x (in the time dependent picture these would be the early time ingoing, positive free fall frequency wavepackets) both annihilate the free fall vacuum $|\text{ff}\rangle$, we derive the following relations as in (62),

$$(a^\dagger(\phi_{+s}) - e^{\pi\omega/\kappa} a(\phi_-^*))|\text{ff}\rangle = 0 \quad (68)$$

$$(a^\dagger(\phi_-^*) - e^{\pi\omega/\kappa} a(\phi_{+s}))|\text{ff}\rangle = 0. \quad (69)$$

These two relations completely determine the ϕ_{+s} and ϕ_-^* content of the free fall vacuum for the Killing frequency ω . Using these relations it is simple to show that the free fall vacuum is a thermal state at the Hawking temperature, exactly as with the ordinary wave equation, [9].

5 Approximate solutions to the superluminal equation

Since the calculations involved in solving the mode equation (12) with the superluminal operator \hat{F}_- are virtually identical to those given above for the subluminal operator \hat{F}_+ , we shall be brief. Computing the approximate solutions outside the horizon by the WKB approximation proceeds exactly as before. The main difference compared to the subluminal case is that we *are* now interested in WKB solutions both inside and outside the horizon. Outside the horizon, the relevant⁸ WKB mode is

$$\phi_{+s}(x) \approx e^{i\omega \int dx/(1+v(x))}, \quad (70)$$

and inside the relevant WKB modes are

$$\phi_{\pm}(x) \approx (-1 + v^2(x))^{-3/4} e^{\pm i \int dx \sqrt{-1+v^2(x)}} e^{i\omega \int dx v(x)/(1-v^2(x))}. \quad (71)$$

As before, it is straightforward to show that these approximate solutions break down around the horizon, but far enough outside the horizon they are valid. To find the appropriate connection formula for these solutions, we now find an approximate solution across the horizon.

Solutions about the horizon can be obtained by again linearizing $v(x)$ as in (18) and solving the resulting approximate mode equation by the method of Laplace transforms. The linearized equation is just (21) with a minus sign inserted before the fourth derivative term. Writing the solution as a Laplace transform as in (22) and substituting into the equation produces the s -space ODE (23) with $s^4 \rightarrow -s^4$. This equation is again trivial to solve. One finds upon writing the solution in the form

$$\phi(x) = \int_C ds g(s) e^{xf(s)} \quad (72)$$

that

$$f(s) = s - \frac{1}{2\kappa x} \left(\frac{s^3}{3} - \frac{i\omega(i\omega - \kappa)}{s} \right) \quad (73)$$

⁸There are other linearly independent solutions both inside and outside the horizon, but they will not be needed in this calculation.

and

$$g(s) = s^{-1-i\omega/\kappa}. \quad (74)$$

Before evaluating (72), first note that at large $|s|$, the integral is dominated by $\exp(-s^3/(6\kappa))$, and therefore for the integral to converge, the contour must asymptote to a region where $\text{Re}(s^3) > 0$. Writing $s = re^{i\theta}$, these regions are given by

$$\text{Region 1} \leftrightarrow \frac{-\pi}{6} < \theta < \frac{\pi}{6} \quad (75)$$

$$\text{Region 2} \leftrightarrow \frac{\pi}{2} < \theta < \frac{5\pi}{6} \quad (76)$$

$$\text{Region 3} \leftrightarrow \frac{7\pi}{6} < \theta < \frac{3\pi}{2}, \quad (77)$$

and are the unmarked regions in figure 5.

To evaluate the contour integral (72), recall from section 3 that our boundary conditions are specified outside the horizon and state that the solution must reduce to a plane wave with wavevector k_{+s} in the constant $v(x)$ region. This solution does not correspond to a saddle point because those solutions are either exponentially growing or decaying. It is not hard to guess what contour we need though, given our past experience with the subluminal dispersion relation. If we take a contour C_6 that encircles the branch cut and asymptotes to regions 2 and 3 (77), see figure 5, we get a contour very similar to the contour C_3 in figure 3. The approximations that went into evaluating that contour also work here. The result is exactly $-\phi_3(x)$ of (36). This solution is just $\phi_{+s}(x)$ (70) up to a multiplicative constant, and therefore the contour C_6 produces the correct boundary condition at $x \gg 0$.

To evaluate (72) for $x \ll 0$ we use the steepest descents approximation. The saddle points in the integrand of (72) are given by

$$s_{\pm} \approx \pm i\sqrt{2\kappa|x|}, \quad (78)$$

and the steepest descent contours must pass through these points in the directions $-\pi/4$ to $3\pi/4$ for s_+ and $\pi/4$ to $5\pi/4$ for s_- . The contours, C_7 and C_8 , therefore must asymptote to regions 1 and 2 and regions 2 and 3 respectively, see figure 6. Furthermore, $C_7 + C_8$ is deformable to C_6 , and therefore is the contour we want. Evaluating (72) over $C_7 + C_8$ by the steepest descents approximation and expressing the result in terms of the WKB solutions (71) results in

$$\phi_7 + \phi_8(x) \approx e^{-i\pi/4}\sqrt{2\pi\kappa}(e^{-\pi\omega/2\kappa}\phi_-(x) - ie^{\pi\omega/2\kappa}\phi_+(x)). \quad (79)$$

We now have the complete solution for all x , which can be displayed as the connection formula

$$e^{-i\pi/4}\sqrt{2\pi\kappa}(e^{-\pi\omega/2\kappa}\phi_-(x) - ie^{\pi\omega/2\kappa}\phi_+(x)) \leftrightarrow \frac{-i2\pi}{\Gamma(1+i\omega/\kappa)}\phi_{+s}(x) \quad (80)$$

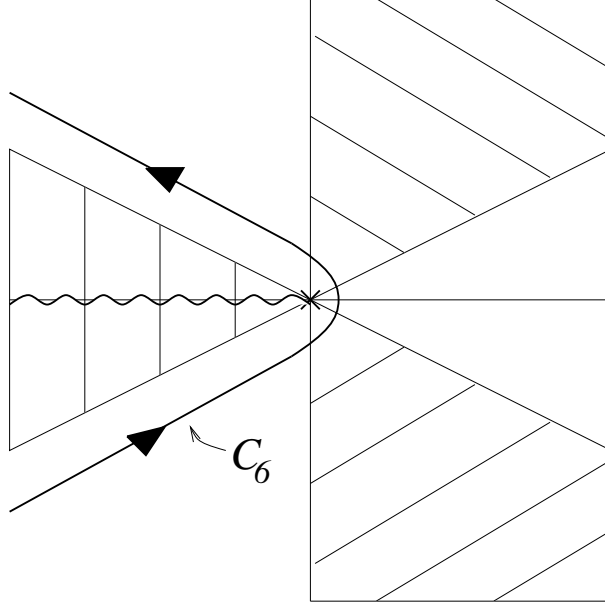


Figure 5: Diagram of the steepest descent contour C_6 . The unmarked regions are directions in which the contour must asymptote for the integral to converge. The \times 's are singularities of the integrand and the wavy line is a branch cut.

where the left-hand-side refers to $x < 0$ and the right-hand-side to $x > 0$. Evaluating (80) at $x \gg 0$ and $x \ll 0$ allows us to pull off the coefficients $c_{+s}(\omega)$ and $c_{-}(\omega)$. Substituting into (14) again produces a thermal spectrum at the Hawking temperature.

As before we have made a number of approximations to arrive at this result. Collecting these approximations together, we can compute the range of validity of these results. However, because the difference between the solutions for the subluminal equation and the superluminal equation is only the change of a few signs in the end, then the difference between the WKB and Laplace transform solutions for the superluminal case is essentially the same as the difference between the WKB and Laplace transform solutions for the subluminal case. Therefore the constraints on the range of validity of the solutions in the parameters κ , x , and ω are the same as in the subluminal case.

5.1 Computing the quantum state

For the superluminal equation we can as well compute the decomposition of the free fall vacuum in terms of particle states as seen by late time observers. We define the out Hilbert space as before, i.e., we take it to be a tensor product of Hilbert spaces inside and outside the horizon respectively. Outside the horizon we define the Hilbert space using Killing frequency, and inside we use free fall frequency (as before we take a $v(x)$ that

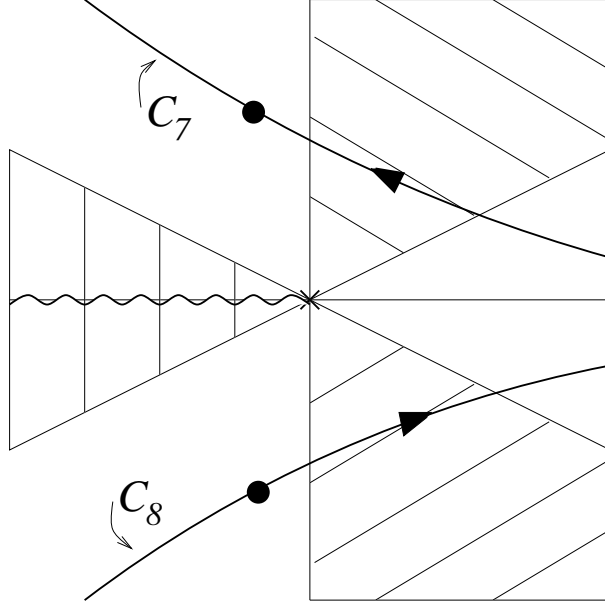


Figure 6: Diagram of the steepest descent contours, C_7 and C_8 , through the saddle points s_+ and s_- respectively. The unmarked regions are directions in which the contour must asymptote for the integral to converge. The \times 's are singularities of the integrand and the wavy line is a branch cut.

asymptotes to a constant smaller than -1 behind the horizon).

To compute the decomposition of the free fall vacuum, we again look for mode solutions which when summed together produce an early time positive free fall frequency packet propagating toward the horizon (but now located behind the horizon). Around the horizon this packet scatters (mode converts) into a pair of packets, a reflected packet which propagates deeper inside the black hole to the constant $v(x)$ region and a transmitted packet which propagates across the horizon out to the constant $v(x)$ region. In this picture the transmitted packet corresponds to the Hawking particle and the reflected packet to the partner. From such a solution we could obtain an equation for the free fall vacuum analogous to (62).

To compute the s-modes (those needed to compute the state) we again first compute the n-modes (those needed to compute number expectation values). We have already computed one n-mode given by (80). Recall that in the time dependent picture this corresponds to a pair of positive and negative free fall frequency packets propagating toward to the horizon (and located inside the black hole) with just the right relative coefficient that the entire packet propagates across the horizon out to the constant $v(x)$ region. The other n-mode therefore corresponds in the time dependent picture to a pair of positive and negative free fall frequency packets propagating toward the horizon with

the right relative coefficient such that the entire packet is reflected and propagates deep inside the black hole to where $v(x)$ is constant. The appropriate mode solution therefore must decay *outside* the horizon.

Computing this mode solution involves the same techniques used already many times, so we shall simply quote the result. The connection formula expressed in terms of the WKB solutions is

$$(e^{-i\pi/4}\sqrt{2\pi\kappa}(-ie^{\pi\omega/(2\kappa)}\phi_+(x) + e^{3\pi\omega/(2\kappa)}\phi_-(x)) + ie^{\pi\omega/\kappa}\frac{2\pi}{\Gamma(1+i\omega/\kappa)}\phi_{-m}(x)) \leftrightarrow e^{-i3\pi/4}e^{\pi\omega/\kappa}\phi_+(x) \quad (81)$$

where the left-hand-side refers to $x < 0$ and the right-hand-side to $x > 0$. The solution $\phi_+(x)$ for positive x decays exponentially with increasing x , and therefore satisfies the boundary conditions.

To compute the s-modes we take linear combinations of the n-modes (80) and (81) to eliminate either $\phi_+(x)$ or $\phi_-(x)$ behind the horizon. This results in the connection formulae

$$\phi_-(x) + iNe^{\pi\omega/\kappa}\phi_{-m}(x) \leftrightarrow iN\phi_{+s}(x) \quad (82)$$

$$\phi_+(x) + N\phi_{-m}(x) \leftrightarrow Ne^{\pi\omega/\kappa}\phi_{+s}(x) \quad (83)$$

where

$$N = e^{i\pi/4}e^{-\pi\omega/(2\kappa)}\frac{\pi}{\sqrt{2\pi\kappa}\sinh(\pi\omega/\kappa)\Gamma(1+i\omega/\kappa)}. \quad (84)$$

These relations are enough to carry out the decomposition of the free fall vacuum as discussed in subsection 4.4. In particular the method of obtaining the equations (69) on the free fall vacuum for the subluminal dispersion relation follows exactly in this case as well, with the replacement of ϕ_- in the subluminal case by ϕ_{-m} in the superluminal case.

6 Conclusions

We have considered two different modifications of the wave equation in a black hole spacetime, one producing subluminal propagation of high frequency modes and the other superluminal propagation of high frequency modes. We have shown that both equations give rise to exactly a thermal spectrum of radiation from a black hole to leading order in an expansion in powers of $1/k_0$. It is natural to try to push the analysis further to obtain a correction term to the outgoing flux. We immediately run into the following difficulty though. In obtaining an approximate solution to (12) about the horizon, we actually solved instead just the linearized equations (21). To obtain a better approximate solution about the horizon, we need a better approximation to (12). We could, for instance, keep

higher order terms in x when expanding $v(x)$ and $v'(x)$. If we try to solve the resulting equation by the Laplace transform method, we find that we get a higher order ordinary differential equation in the Laplace transform variable s . In other words, the s -space equation is really not much better than the original x -space equation.

An important assumption made in deriving the thermal radiation for the superluminal equation of motion was that positive free fall frequency modes, located behind the horizon, were in their ground state. Clearly we don't know a priori whether this is the physically correct quantum state condition. In principle we would have to begin with a quantum state which evolves into a black hole, and then ask if these modes actually are in their ground state. This requires quantum gravity. A more realistic problem to tackle at this time is simply to ask where these modes came from in a semiclassical approximation. One would guess naively from the singularity. Recent investigations, [7, 12], have shown; however, that for certain charged black holes it is possible that these modes simply reflect outside the singularity and become ingoing modes, backward in time. This would have important implications because it would mean that the Hawking radiation, even for an eternal black hole, would originate from ingoing modes, and therefore we would not have the infinite degrees of freedom problem, [7].

We end by noting that the subluminal model (and possibly the superluminal model as well) considered in this paper suffers from the "stationarity puzzle". If we try to propagate the outgoing modes backward in time all the way out to infinity, where $v(x)$ goes to zero, then there can be no particle creation by conservation of Killing frequency. One way out of this problem is to introduce time dependence into the equation of motion (perhaps via backreaction) to destroy the Killing symmetry. A step in this direction is to put the ordinary wave equation on a spatial lattice, this has the advantage of introducing naturally a short distance cutoff and at the same time destroying the Killing symmetry (for discretizations of most spatial coordinates). Such a model is currently being investigated [18] by techniques similar to those described in this paper.

7 Acknowledgements

I would like to thank Ted Jacobson for many valuable conversations and suggestions on a draft of this paper, and also Jorma Louko for helpful discussions. This work was supported in part by NSF grant PHY94-13253.

References

- [1] S.W. Hawking, *Nature (London)* **248** (1974) 30; *Commun. Math. Phys.* **43** (1975) 199.

- [2] J. B. Hartle and S. W. Hawking, Phys. Rev. **D13** (1976) 2188.
- [3] W. G. Unruh, Phys. Rev. **D14** (1976) 870.
- [4] K. Fredenhagen and R. Haag, Commun. Math. Phys. **127** (1990) 273.
- [5] W.G. Unruh, Phys. Rev. D **51** (1995) 2827.
- [6] R. Brout, S. Massar, R. Parentani and Ph. Spindel, Phys. Rev. D **52** (1995) 4559.
- [7] T. Jacobson, Phys. Rev. D **53** (1996) 7082.
- [8] S. Corley and T. Jacobson, Phys. Rev. D **54** (1996) 1568.
- [9] R. M. Wald, Commun. Math. Phys. **45** (1975) 9.
- [10] W. G. Unruh, personal communication.
- [11] S. Corley and T. Jacobson, work in progress.
- [12] G. T. Horowitz and D. Marolf, Phys. Rev. D **52** (1995) 5670.
- [13] T. Jacobson, “Semiclassical Decay of Near-Extremal Black Holes”, UMDGR-97-110 [hep-th/9705017].
- [14] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1968).
- [15] N. Bleistein and R. A. Handelsman, *Asymptotic Expansions of Integrals* (Holt, Rinehart and Winston, New York, 1975).
- [16] R. Wong *Asymptotic Approximations of Integrals* (Academic Press, New York, 1989).
- [17] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, New York, 1994).
- [18] S. Corley and T. Jacobson, “Lattice black holes”, UMDGR-98-23 [hep-th/9709166].